FUEL OPTIMAL IN ORBIT POSITION ASSIGNMENT OF FORMATION FLYING SPACECRAFT

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Abstract: This work addresses the problem of fuel optimal control of a group of spacecraft to pre-defined relative formations. We consider relative motion of spacecraft on a circular orbit based on the Clohessy-Wiltshire equations. We find the optimal assignment between the set of spacecraft and the desired formation shape such that each spacecraft reaches its final position in specified time while minimizing the total ΔV required by all spacecraft. The resulting algorithm is computationally efficient and suitable for large number of spacecraft in formation.

Keywords: Formation Assignment, Fuel Optimal, Hungarian Algorithm.

1. Introduction

Distributed space systems are subject of a number of planned missions and promise to revolutionize use of outer space as we know it. Distributed space systems rely on a number of - usually - smaller and lighter spacecraft that fly in the vicinity of each other and the collection of spacecraft fulfill the mission objective. This architecture is more robust to failure of a single spacecraft and is perceived to be easier to launch. However, a number of technical challenges need to be addressed before distributed spacecraft missions can become reality. One of these technical issues is to devise an efficient algorithm for translating the formation flying spacecraft to achieve the desired relative formation and maintain that formation. In particular, this task becomes more ambitious as the number of spacecraft increase to thousands as some of the proposed applications suggests.

Problem of formation assignment of a large number of pico-satellites is of particular interest to mission designers in distributed spacecraft community. From a practical point of view, algorithms used for this purpose should be computationally efficient, distributable, and rely on least amount of information possible. These attributes allows for the algorithm to run on-board few hundred gram size spacecraft with very limited computational capacity.

The problem of spacecraft formation reconfiguration first was introduced by Wang and Hadaegh [8]. They precisely defined reconfiguration and reduced the problem to study of the permutation groups. Richards *et al.* used Mixed Integer Linear Programming (MILP) to determine fuel optimal spacecraft trajectories that avoid obstacles [10]. The MILP framework also allows spacecraft to be optimally assigned to a set of target positions when all of them are the same. A similar approach to optimal assignment - to a rotated version of the desired formation map - based on Lee Algebra and combinatorial techniques is proposed by Lee *et al.* [5] . Acikmese *et al.* introduced an optimal formulation based on the Second Order Cone Programming (SOCP), that could avoid

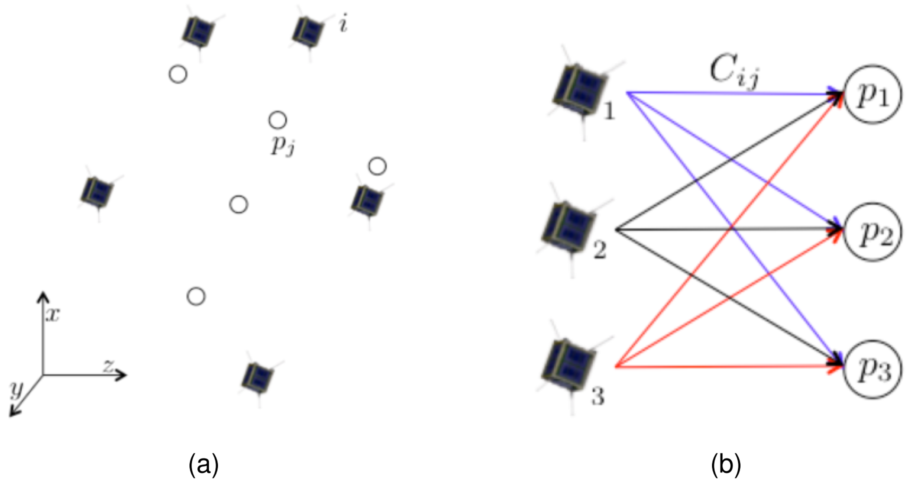


Figure 1: (a) Current (cube) and desired (circle) position of spacecraft in Hill's frame (b) Assignment problem can be expressed as a bipartite graph with weight C_{ij} on an edge between spacecraft i and position j.

obstacles and produce optimal path between a spacecraft formation and desired position of each one of them [1]. A set of proposed formation flying algorithms achieve the desired formation by using virtual structures in term of rigid graphs and by specifying the position of each spacecraft in the formation [9].

In the robotics community, similar approaches are used to optimally assign assets to desired positions. A number of work by Zavlanos and Pappas, such as [13], take advantage of market based algorithms, where robots bid on prospective assignments, to achieve a decentralized solution. Potential functions are also used to guide the robots to the next unoccupied formation position [12]. Another popular approach takes advantage of the consensus algorithms by defining formation as a set of desired inter-agent distances [7]. Macdonald using Kuhn's algorithm suggested a method to minimize the collective distance between current robot positions and a translated and rotated destination map [6].

Inspired by the above body of work, this paper establishes an efficient algorithm for spacecraft formation assignment to a desired formation map, while subject to the orbital dynamics.

2. Spacecraft Formation Assignment Problem

We focus on the problem of assigning positions in a specified formation map to individual spacecraft such that the combined effort of all spacecraft for going from current state to the desired final state (formation map) is minimized while spacecraft movements are subject to orbital dynamics. Figure 1(a) illustrates this concept.

This problem, as depicted by Figure 1(b), can be represented as a *directed bipartite graph* with edges from each spacecraft to each of the desired positions. Each edge has a weight C_{ij} representing the cost of spacecraft i being assigned to position p_j. Defining π as mapping between spacecraft and desired position, i.e. $\pi : x_i \to p_j$, the

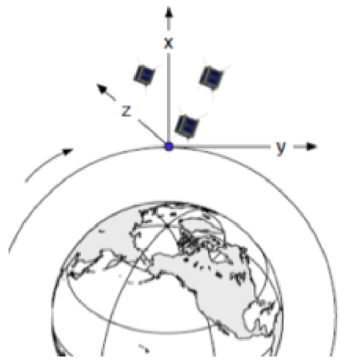


Figure 2: Hill's reference frame, centered on a point on reference circular orbit with radial (x), alongtrack (y), and out of plane (z) directions.

optimal assignment is defined as

$$\pi^* = \underset{\pi}{\operatorname{argmin}} \sum_i C_{x_i,\pi(x_i)}. \tag{1}$$

Our goal is to devise a distributed algorithm that reduces the complexity of assignment problem of (1) from $n!$ to polynomial time. Next section will describe how the cost of transformation from x_i to p_j can be estimated while taking the orbital dynamics of the spacecraft into account.

3. Orbital Maneuvers

We use the Clohessy-Wiltshire (CW) equations of linear motion of spacecraft with respect to a point on the desired circular orbit [11]. Clohessy-Wiltshire or Hill's equations (as sometimes referred to) describe spacecraft relative motion in a rotating frame centered on a target spacecraft on the desired orbit with x-axis in the radial direction, y-axis in the along-track direction and z-axis completing the right hand frame and pointing in the out of plane direction. Figure 2 depicts the Hill's frame.

The CW equation for the motion of the i-th spacecraft is of the form

$$\dot{x}_i = Ax_i + B_i u_i \tag{2}$$

where x_i represents the states (position and speed) of the i-th spacecraft. State and control matrices, A and B respectively, are of the form

$$\begin{aligned} A &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3\omega^2 & 0 & 0 & 0 & 2\omega & 0 \\ 0 & 0 & 0 & -2\omega & 0 & 0 \\ 0 & 0 & -\omega^2 & 0 & 0 & 0 \end{bmatrix}, \\ B_i &= \frac{1}{m_i} \begin{bmatrix} \mathbf{0}_{3\times3} \\ I_{3\times3} \end{bmatrix}. \end{aligned}$$

Natural frequency of the reference orbit $\omega = \sqrt{\frac{GM_e}{Re+h}^3}$; where G is the gravitational constant, M_e is mass of the Earth, R_e refers to the average Earth radius, and h is the altitude of the orbit. This orbis has a period of $T = 2\pi/\omega$.

Clohessy-Wiltshire equation has a homogeneous solution ($u_i = 0$) of the form

$$x_i(t) = e^{At} x_i(0). \tag{3}$$

Matrix exponential e^{At} has an analytical presentation as

$$e^{At} = \begin{bmatrix} 4-3C & 0 & 0 & S/\omega & -(2C+2)/\omega & 0 \\ 6S-6\omega t & 1 & 0 & (2C-2)/\omega & (4S-3\omega t)/\omega & 0 \\ 0 & 0 & C & 0 & 0 & S/\omega \\ 3\omega S & 0 & 0 & C & 2S & 0 \\ 6\omega C-6\omega & 0 & 0 & -2S & -3+4C & 0 \\ 0 & 0 & -\omega S & 0 & 0 & C \end{bmatrix}, \tag{4}$$

where $C = \cos(\omega t)$ and $S = \sin(\omega t)$.

3.1. Two Burn Maneuvers

Considering impulse thrusters for formation maneuvers, Equation (3) is used to calculate ΔV required to move from a relative state in the Hill's frame to another relative state. Let us assume we intend to move from $x(0) = [\, r_0^T \quad v_0^T \,]^T$ to $x(t_f) = [\, r_f^T \quad v_f^T \,]^T$ using a two-burn maneuver. Variables r and v represent position and velocity respectively. Partitioning e^{At} into four 3×3 matrices, state of spacecraft after a period of t_f seconds according to equation (3) is calculated as

$$\begin{bmatrix} r_f \\ v_f^- \end{bmatrix} = \begin{bmatrix} \Phi_{rr} & \Phi_{rv} \\ \Phi_{vr} & \Phi_{vv} \end{bmatrix} \Bigg|_{t=t_f} \begin{bmatrix} r_0 \\ v_0^+ \end{bmatrix}. \tag{5}$$

Here, v_0^+ is the speed of the spacecraft after the initial burn and v_f^- is the speed of spacecraft after arrival at its final destination r_f and before the final burn.

From (5), v_0^+ and v_f^- are found to be

$$\begin{aligned} v_0^+ &= \Phi_{rv}^{-1} \left(r_f - \Phi_{rr} r_0 \right), & (6) \\ v_f^- &= \Phi_{vr} r_0 + \Phi_{vv} v_0^+. & (7) \end{aligned}$$

Hence, ΔV required for this maneuver is calculated as

$$\Delta V(x_0, x_f) = \overbrace{\|v_0^+ - v_0\|}^{\Delta V_0} + \overbrace{\|v_f - v_f^-\|}^{\Delta V_f}. \tag{8}$$

3.2. Optimal Formation Keeping

Another potential way of controlling a spacecraft from an initial state to a desired relative state (or potentially keeping the spacecraft at the desired state) using continuous thrusters is to use optimal control techniques; In particular, Linear Quadratic Regulators (LQR). Objective in this case is to find control input $u_i(t)$ such that it minimizes a quadratic cost function on state error and control effort. i.e.,

$$\min_{u_i(t)} \mathcal{J} = \int_0^{t_f} \Big(e_i(t)^T Q \, e_i(t) + u_i(t)^T R \, u_i(t) \Big) dt, \tag{9}$$

with $e_i(t) = x_i(t) - x_i(t_f)$; subject to dynamics of (2).

Matrices $Q_{6\times6} \succeq 0$ and $R_{3\times3} \succ 0$ are weighting matrices, typically chosen to provide a balance between the position error and control effort. A good first choice for these matrices is an appropriate weight multiplied by an identity matrix of proper size.

Solution of LQR problem is well studied [2]. The optimal control $u_i^*(t) = K_i(t)e_i(t)$, is computed from optimal gain $K_i(t) = -R^{-1}B^TP_i(t)$, where $P(t)$ is the solution of the Riccati equation

$$-\dot{P}(t) = A^TP(t) + P(t)AP(t)BR^1B^TP(t) + Q.$$

4. Optimal in-orbit formation assignment

As mentioned before, goal is to find an optimal assignment between current formation positions and desired positions in Hill's frame such that the overall fuel consumption of all space craft - represented by ΔV required - is minimized.

Given a set of desired positions in Hill's frame p_j, and a set of initial spacecraft states $x_i = [\, r_i^T \;\, v_i^T \,]^T$, the assignment cost, using (8), is defined as

$$C_{ij} = \Delta V\left(x_i, \begin{bmatrix} p_j \\ 0 \end{bmatrix}\right). \tag{10}$$

Alternately this cost can be calculated using the optimal control results as $C_{ij} = \int_0^{t_f} u_i(t)dt$, where $u_i(t)$ is the solution of (9) with $x_i(t_f) = \begin{bmatrix} p_j \\ 0 \end{bmatrix}$.

The assignment problem of (1) is equivalent to finding an assignment matrix $\mathcal{X}$, with entries $\mathcal{X}_{ij} = 1$ when spacecraft i is assigned to position j and zero otherwise. Hence

$$\mathcal{X}^* = \operatorname*{argmin}_{\mathcal{X}} \sum_i \sum_j \mathcal{X}_{ij}C_{ij}.$$

Inspired by work of Macdonald [6], we use Kuhn's Hungarian algorithm [4] for solving the assignment problem subject to orbital dynamics. The Hungarian algorithm is often used to find maximum matching in an appropriate bipartite graph such as the one depicted in Figure 1(b). This algorithm reduces the complexity of the problem from $n!$, worse case scenario, to an efficient polynomial time $\mathcal{O}(n^3)$ [3]. Runtime of our proposed algorithm as a function of number of spacecraft for randomly generated initial conditions is illustrated in Figure 3.

So far, we addressed the centralized assignment. However, the same scheme can be used to converge to the optimal solution in a decentralized fashion. In reality, the polynomial assignment time allows each spacecraft to solve their own version of the assignment problem. We assume set of all desired positions and states of all spacecraft in formation is available to each spacecraft through distribute estimation. Since, all spacecraft solve the assignment problem based on the same common data, their solutions should converge to the same assignment.

Throughout the maneuver (after the first burn, and before the final burn), and provided no external disturbances are present, total ΔV required to reach the desired positions in the remaining time is zero. Hence, in the absence of external purturbations or estimation errors the optimal assignment will not change during the Maneuver. Hence, the assignment routine can continuously run in the background to provide feedback and allow to adapt to any external disturbances.

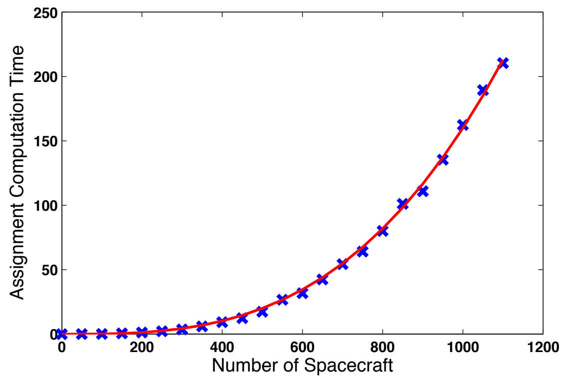


Figure 3: Simulation runtime of assignment algorithm

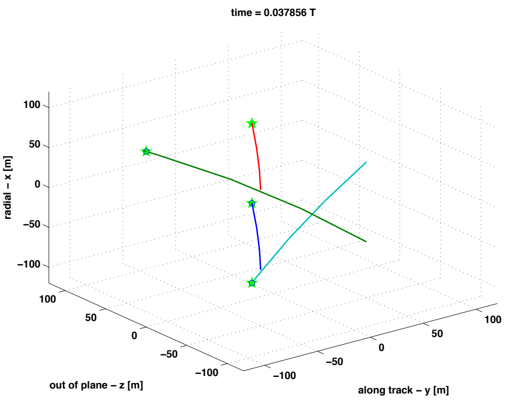
5. Simulations

Figure 4 demonstrates an optimal assignment and reconfiguration maneuver for a set of four spacecraft in a 402 Km circular orbit (similar to ISS), with a period of $T = 92.5$ minutes. Throughout the simulation spacecraft formation morph form its current state to the desired map taking advantage of the orbital dynamics to minimize their collective effort by assigning most suitable spacecraft to each position. When spacecraft arrive at the desired location, optimal feedback control of $\S$3.2, will keep the spacecraft in desired formation until the next target formation map is set.

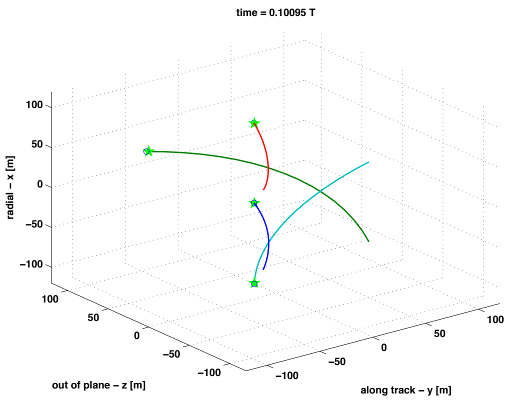
As seen in various subfigures, for the same set of initial conditions and desired formation map, optimal assignment can vary based on the length of the desired maneuver, t_f. This variation is due to the dynamics of CW framework; Very short maneuver times dictate spacecraft to move towards its assigned destination almost in a straight line, while longer maneuver times allow the spacecraft to take maximum advantage of the orbital dynamics to reduce the required effort. However, very long maneuver times might not offer better solutions, as spacecraft choose longer paths only to comply with the final time constraint.

6. Conclusions

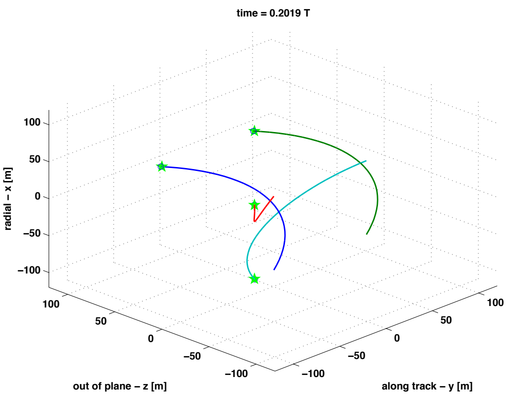
The proposed formation reconfiguration algorithm take advantage of a computationally efficient framework to assign spacecraft to a desired formation map. The cost associated to each association can be modified so that it address many factors the assignment is desired to be optimal with respect to them. Here we defined this cost as ΔV required for a two burn maneuver that takes each spacecraft from current location to any of the desired positions. Algorithm assumes each spacecraft has perfect knowledge of other spacecraft's relative states. This assumption for very large number of spacecraft (large swarms) can be a limiting factor. However, using consensus based estimation algorithms this requirement can be relaxed since the estimated state will eventually converge to the state of all the spacecraft in formation (even when no direct measurement is not provided). The optimality of assignment allows for the proposed formation assignment algorithm to run in the background, hence the assignment will converge to the optimal assignment as the estimated states approach the actual states of the spacecraft.



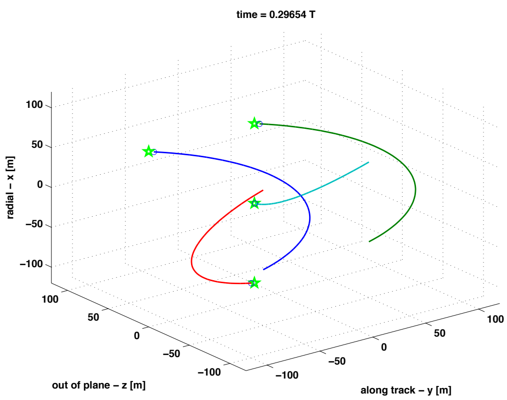
(a) $t_f = .02T$



(b) $t_f = .1T$



(c) $t_f = .2T$



(d) $t_f = .3T$

Figure 4: Optimal assignments can change with the length of the maneuver.

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